Deformation quantization and $\exp(\frac{i}{\hbar})$

For a function f on a manifold X, consider the twisted De Rham complex $(\Omega^*(X), d_f = d_{\mathrm{DR}} + \frac{1}{i\hbar} df)$. If $\hbar$ is a nonzero number, d_f is just the conjugation of d by $\exp(\frac{1}{i\hbar} f)$; therefore its cohomology is the De Rham cohomology. If $\hbar$ is a formal parameter, and if one replaces $\Omega^*(X)$ by $\Omega^*(X)((\hbar))$, the cohomology of the twisted De Rham complex is not the De Rham cohomology. For example, if f is a Morse function, then the cohomology is concentrated in degree dim(X), and its basis over $\mathbb{C}((\hbar))$ is indexed by critical points of f. The twisted De Rham complex can be interpreted as $\mathrm{RHom}_A(V_0, V_f)$ where A is the algebra of $\hbar$-differential operators on X and V_0, V_f are two modules over this algebra. In another, related version, A is the canonical deformation quantization of T^*X. (The twisted De Rham complex has several other important cohomological descriptions). In mid-eighties, Witten showed how one recovers critical points of f, together with their Morse indices, from the asymptotic analysis of eigenvalues of the Laplacian constructed from d_f. At about the same time, Boris Feigin asked whether one can recover the same information from deformation quantization of symplectic manifolds and relate that to the theory of Lagrangian intersections. In this talk, we propose a solution to this question. We define a new category, consisting of modules over deformation quantization on which expressions of the form $\exp(\frac{1}{i\hbar} f)$ act. We explain what this means, using the example of the symplectic manifold $\mathbb{R}^{2n}$. This work is related to Dima Tamarkin's work on microlocal methods in symplectic topology.